Due Sat.

(a.k.a. endpoint problems)

5.2 - Linear Models: Boundary-Value Problems

Consider $y'' + 3y = 0,\ y(0) = 0,\ y(\pi) = 0$ different values here

Trivial solution $y = 0$.

We can find that $y = a\cos(\sqrt{3}x) + b\sin(\sqrt{3}x)$

since $y(0) = 0$, $a = 0 \Rightarrow y = b\sin(\sqrt{3}x)$

$y(\pi) = 0 \Rightarrow 0 = b\sin(\sqrt{3}\pi) \Rightarrow b = 0$

We only have the trivial solution.

Now consider $y'' + 4y = 0,\ y(0) = 0,\ y(\pi) = 0$

$y = a\cos 2x + b\sin 2x$ $\quad a = 0$

$y = b\sin 2x$ with $y(\pi) = 0$

$0 = b\sin(2\pi) \Rightarrow b \in \mathbb{R}$

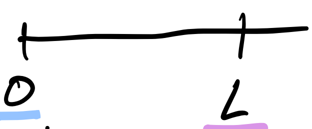
Thus $y = b\sin 2x$ is a solution to the BVP for $b \in \mathbb{R}$. The BVP has an infinite number of solutions.

lambda

Definition: A number λ for which there exist nontrivial solutions to $y'' + P(x)y' + \lambda Q(x)y = 0,\ y(a) = 0,\ y(b) = 0$ is called an **eigenvalue**, and a solution $y = ay_1 + by_2$ that satisfies the BVP is an **eigenfunction** associated with the eigenvalue λ.

In the previous example, $P(x) = 0$, $Q(x) = 1$
$\lambda = 4$ is the eigenvalue, and
$y = b \sin 2x$ is the eigenfunction.
$y'' + 3y = 0$, $y(0) = 0$, $y(\pi) = 0$ has no eigenvalue.
We generalize this:

Example: Determine eigenvalues and eigenfunctions for
$y'' + \lambda y = 0,\ y(0) = 0,\ y(L) = 0\ (L > 0)$.



Case I: $\lambda = 0$. Then $y = ax + b$
The boundary conditions give $a = b = 0$
We only have the trivial soln. No eigenvalue.

Case II: $\lambda < 0$. Let $\lambda = -\alpha^2$

$$y'' - \alpha^2 y = 0 \qquad m^2 - \alpha^2 = 0$$
$$m = \pm \alpha$$

$$y = a e^{\alpha x} + b e^{-\alpha x}$$

Side bar: $\left(\cosh x = \frac{e^x + e^{-x}}{2}, \ \sinh x = \frac{e^x - e^{-x}}{2} \right)$

so we can write $y = a\cosh \alpha x + b \sinh \alpha x$

$y(0) = 0 \Rightarrow a = 0$, $y(L) = 0 \Rightarrow b = 0$

Again we only have the trivial solution

No eigenvalue here.

Case III: $\lambda > 0$, say $\lambda = \alpha^2$

$y'' + \alpha^2 y = 0 \Rightarrow m = \pm \alpha i$

$y = a\cos \alpha x + b \sin \alpha x$

$y(0) = 0 \Rightarrow a = 0$, and $y(L) = 0$

yields $b \sin \alpha L = 0$

This is zero whenever $\alpha L = n\pi, n \in \mathbb{Z}^+$

Assuming $\alpha > 0$ and $L > 0$,

$$\alpha = \frac{n\pi}{L} \Rightarrow \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}$$

We have an infinite sequence of eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \dots$$

Consider a vertical beam of length L with load P applied to one end or a road with compressive force P applied to one end. Theory of elasticity provides the di□erential equation

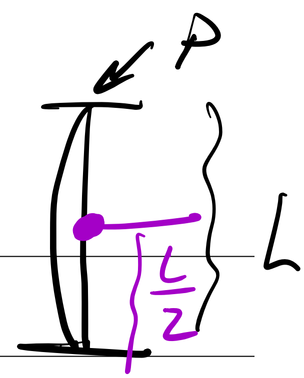
$EI\dfrac{d^2y}{dt^2} = -Py$ or $EI\dfrac{d^2y}{dt^2} + Py = 0$ which in standard form is

$$\frac{d^2y}{dt^2} + \frac{P}{EI}y = 0 \qquad (y'' + \lambda y = 0)$$

The product EI is the flexural rigidity of the beam, E is Young's modulus of elasticity, and I is the moment of inertia of a cross-section of the beam.

Considering $P(x) = 0,\ Q(x) = 1$, and letting $\lambda = \dfrac{P}{EI}$, this has the form

$$y'' + \lambda y = 0$$



With the substitution $\lambda = \frac{P}{EI}$ and considering $y(0)=0$, $y(L)=0$, we have solutions $y = b\sin\left(\frac{n\pi x}{L}\right)$ so the column/road will buckle/deflect if $P = \dfrac{n^2\pi^2 EI}{L^2}$. When $n=1$, we have the first buckling load.

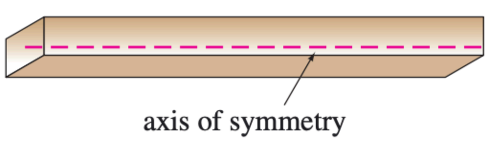
If we restrain/support the system at $L/2$, then the smallest critical (buckling) load will be $P = \dfrac{4\pi^2 EI}{L^2}$

Beams:

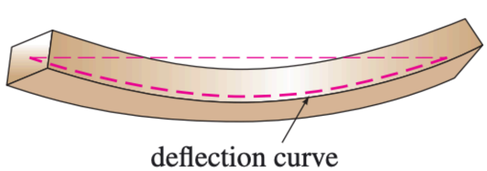
Considering a beam of length L, the bending moment (response or resistance to load) $M(x)$ is related to the load per unit length $w(x)$ by $\dfrac{d^2M}{dx^2} = w(x)$ (this is from theory of elasticity). Assuming down is positive, $M(x) = EI\kappa$ is proportional to the curvature of the elastic curve. From multivariable calculus, curvature is $\kappa = \dfrac{y''}{[1+(y')^2]^{3/2}}$. For small deflection, $y' = 0$, so $M = EIy''$. We see that $M'' = w(x)$ becomes

$$EI\frac{d^4y}{dx^4} = w(x)$$

Definition: The graph of the function $y(x)$ is the **deflection curve** of the beam.

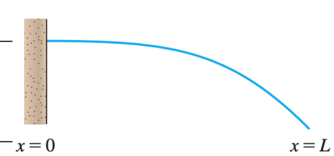


(a)

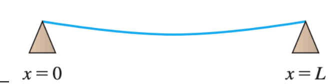


(b)

(a) embedded at both ends

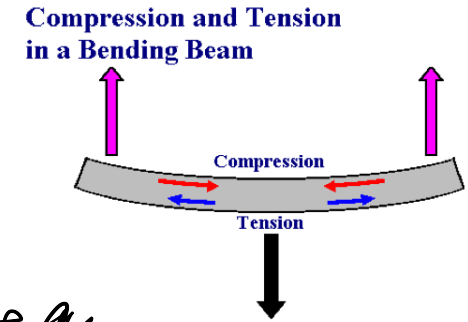


(b) cantilever beam: embedded at the left end, free at the right end



(c) simply supported at both ends

For the deflection $y(x)$, we have the following:
y' is the slope of the curve;
y'' is the bending moment;
y''' is the shear force.



If the end of the beam
is embedded, $y=0$, $y'=0$
free $y''=0$, $y'''=0$
Simply supported/hinged $y=0$, $y''=0$

Example: Solve the di□erential equation $EI\dfrac{d^4y}{dx^4} = w(x)$ subject to the appropriate boundary conditions. The beam is of length L, and w_o is a constant.

(a) The beam is simply supported at both ends, and $w(x) = w_0$, $0 < x < L$.

$$y^{(4)}(x) = \frac{w_0}{EI} \Rightarrow y'''(x) = \frac{w_0}{EI}x + C_1$$

$$y'' = \frac{w_0}{2EI}x^2 + C_1 x + C_2$$

$$y' = \frac{w_0}{6EI}x^3 + \frac{1}{2}C_1 x^2 + C_2 x + C_3$$

$$y = \frac{w_0}{24EI}x^4 + \frac{1}{6}C_1 x^3 + \frac{1}{2}C_2 x^2 + C_3 x + C_4$$

Simply supported at $x=0$ and $x=L$ means

$$y(0) = y(L) = 0 \,, \quad y''(0) = y''(L) = 0$$

$\hookrightarrow c_4 = 0$ $\qquad\qquad$ $\hookrightarrow c_2 = 0$

$$y(x) = \frac{w_0}{24EI} x^4 + \frac{1}{6} c_1 x^3 + c_3 x$$

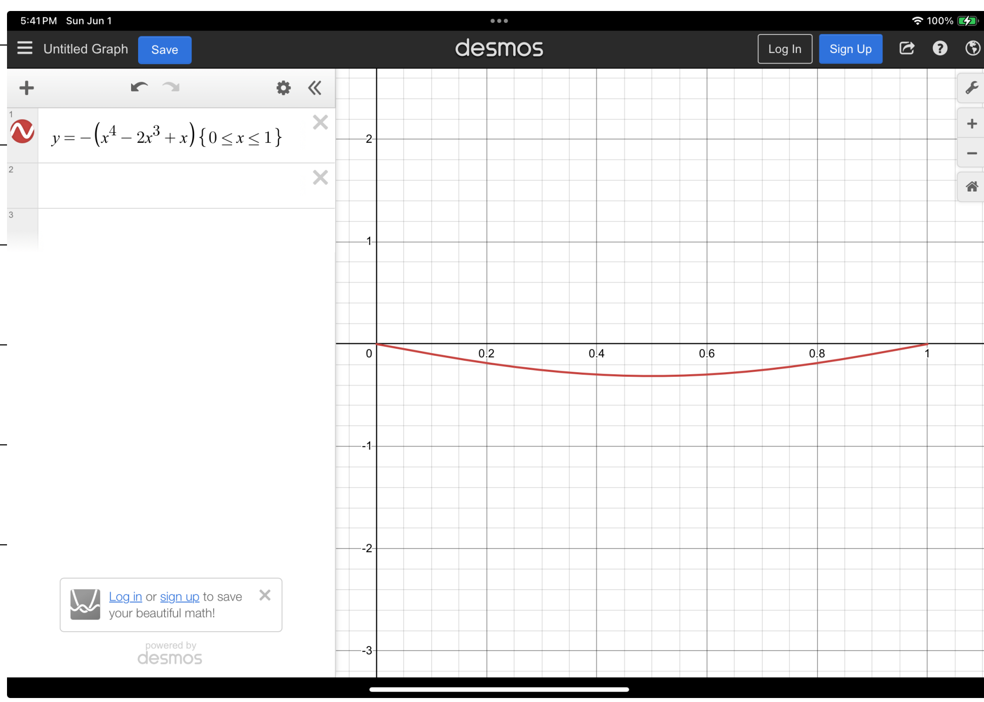
$$y(L) = 0 \Rightarrow 0 = \frac{w_0}{24EI} L^4 + \frac{1}{6} c_1 L^3 + c_3 L$$

$$y''(L) = 0 : 0 = \frac{w_0}{2EI} L^2 + c_1 L \longrightarrow c_1 \nearrow \quad \hookrightarrow c_3$$

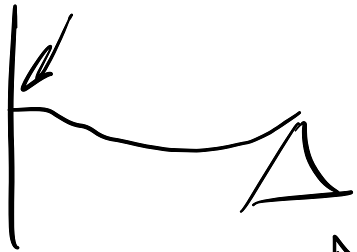
we find that $\boxed{y(x) = \frac{w_0}{24EI}\left(x^4 - 2Lx^3 + L^3 x\right)}$

(b) Use a graphing utility to graph the deflection curve when $w_0 = 24EI$ and $L = 1$.

$$y(x) = x^4 - 2x^3 + x, \quad 0 \le x \le 1$$



embedded: $y(0)=0,\ y'(0)=0$



simply supported

$$y(L)=0,\quad y''(L)=0$$

$$y^{(4)} = \frac{w_0}{EI}$$

$\downarrow$

y''' $+c_1$

$\downarrow$

y'' $+c_2$

$\downarrow$

y' $+c_3$

$\downarrow$

y $+c_4$

$$\frac{y(0),\ y'(0),\ y''(0),\ y'''(0)}{y(L),\ y'(L),\ y''(L),\ y'''(L)}$$